

# ABC AND THE HASSE PRINCIPLE FOR QUADRATIC TWISTS OF HYPERELLIPTIC CURVES

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ABSTRACT. Conditionally on the ABC conjecture, we show that a hyperelliptic curve  $C/\mathbb{Q}$  of genus at least three has infinitely many quadratic twists that violate the Hasse Principle iff it has no  $\mathbb{Q}$ -rational hyperelliptic branch points.

## 1. INTRODUCTION

Let  $C/\mathbb{Q}$  be an algebraic curve. (All our curves will be *nice*: smooth, projective and geometrically integral.) An involution  $\iota$  on  $C$  is an order 2 automorphism of  $C/\mathbb{Q}$ . For any quadratic field  $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ , there is a curve  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$ , the quadratic twist of  $C$  by  $\iota$  and  $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ . After extension to  $\mathbb{Q}(\sqrt{d})$  the curve  $\mathcal{T}_d(C, \iota)$  is canonically isomorphic to  $C_{/\mathbb{Q}(\sqrt{d})}$ , but the  $\text{Aut}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) = \langle \sigma_d \rangle$  action on  $C(\mathbb{Q}(\sqrt{d}))$  is “twisted by  $\iota$ ”:  $\sigma : P \in C(\mathbb{Q}(\sqrt{d})) \mapsto \iota(\sigma_d(P))$ , and thus

$$\mathcal{T}_d(C, \iota)(\mathbb{Q}) = \{P \in C(\mathbb{Q}(\sqrt{d})) \mid \iota(P) = \sigma_d(P)\}.$$

If  $d \in \mathbb{Q}^{\times 2}$  we put  $\mathcal{T}_d(C, \iota) = C$ , the “trivial quadratic twist.”

Let  $q : C \rightarrow C/\iota$  be the quotient map. Every  $\mathbb{Q}$ -rational point on  $\mathcal{T}_d(C, \iota)$  maps via  $q$  to a  $\mathbb{Q}$ -rational point on  $C/\iota$ . Let  $\bar{P} \in C/\iota(\mathbb{Q})$ . If  $\bar{P}$  a branch point of  $\iota$ , then the unique point  $P \in C(\mathbb{Q})$  such that  $q(P) = \bar{P}$  is also rational on every quadratic twist. If  $\bar{P}$  is not a branch point of  $\iota$ , there is a unique  $d \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$  such that the fiber of  $q : \mathcal{T}_d(C, \iota) \rightarrow C/\iota$  consists of two  $\mathbb{Q}$ -rational points.

Work of Clark and Clark-Stankewicz [Cl08], [ClXX], [CS18] gives criteria on  $C$  and  $\iota$  for there to be infinitely many  $d \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle: letting  $\mathbf{A}_{\mathbb{Q}}$  be the adèle ring over  $\mathbb{Q}$ , this means  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$  but  $\mathcal{T}_d(C, \iota)(\mathbb{Q}) = \emptyset$ . Here is one version.

**Theorem 1.** [CS18, Thm. 2] *Let  $C/\mathbb{Q}$  be a nice curve, and let  $\iota$  be an involution on  $C$ . Suppose:*

- (T1) *The involution  $\iota$  has no  $\mathbb{Q}$ -rational branch points.*
- (T2) *The involution  $\iota$  has at least one geometric branch point:  $\{P \in C(\overline{\mathbb{Q}}) \mid \iota(P) = P\} \neq \emptyset$ .*
- (T3) *For some  $d \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$  we have  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ .*
- (T4) *The set  $(C/\iota)(\mathbb{Q})$  is finite.*

*Then, as  $X \rightarrow \infty$ , the number of squarefree  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_C \frac{X}{\log X}$ .*

An involution  $\iota$  on a curve  $C/\mathbb{Q}$  is hyperelliptic if  $C/\iota \cong \mathbb{P}^1$ . A hyperelliptic curve is a pair  $(C, \iota)$  with  $\iota$  a hyperelliptic involution on  $C$ . (A curve of genus at least two admits at most one hyperelliptic involution.) A hyperelliptic curve  $(C, \iota)$  of genus  $g$  has an affine model  $y^2 = f(x)$  with  $f(x) \in \mathbb{Q}[x]$  squarefree of degree  $2g+2$  and  $\iota : (x, y) \mapsto (x, -y)$ . The twist  $\mathcal{T}_d(C, \iota)$  has affine model  $dy^2 = f(x)$ . The branch points of  $\iota$  are the roots of  $f$  in  $\overline{\mathbb{Q}}$ .<sup>1</sup>

If  $\iota$  is a hyperelliptic involution then  $(C/\iota)(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$  is infinite, so (T4) is not satisfied. In this note we give a *conditional* complement to Theorem 1 that applies to hyperelliptic curves.

<sup>1</sup>We have chosen a model in which the point at  $\infty$  is not a branch point; this is always possible. There is a model in which the point at  $\infty$  is a branch point iff there is a  $\mathbb{Q}$ -rational branch point.

**Theorem 2.** *Assume the ABC conjecture. For a hyperelliptic curve  $(C, \iota)$  of genus  $g \geq 3$ , the following are equivalent:*

- (i) *The hyperelliptic involution  $\iota$  has no  $\mathbb{Q}$ -rational branch points.*
- (ii) *As  $X \rightarrow \infty$ , the number of squarefree integers  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_C \frac{X}{\log X}$ .*
- (iii) *Some quadratic twist  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle.*

Certainly (ii)  $\implies$  (iii). As for (iii)  $\implies$  (i): if  $\iota$  has a  $\mathbb{Q}$ -rational branch point then this point stays rational on every quadratic twist.

The crux is to show that (i)  $\implies$  (ii), which we will do in §2. In §3 we give upper and, in a special case, lower bounds on the number of quadratic twists having adelic points locally, leading to the conclusion that when hyperelliptic curves of genus  $g \geq 3$  are ordered by height, for 100% of such curves the number of twists up to  $X$  that violate the Hasse Principle is  $o(X)$ , but that conditionally on ABC, there is a class of hyperelliptic curves for which the number of twists up to  $X$  that violate the Hasse Principle is  $\gg X$ . Some final remarks are given in §4.

## 2. PROOF OF THEOREM 2

### 2.1. Local.

**Theorem 3.** *Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g \geq 1$ . If  $C(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ , then the set of primes  $p \equiv 1 \pmod{8}$  for which  $\mathcal{T}_p(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$  has positive density.*

*Proof.* For any place  $\ell \leq \infty$  of  $\mathbb{Q}$ , if  $p \in \mathbb{Q}_{\ell}^{\times 2}$  then  $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_{\ell}} \cong C_{/\mathbb{Q}_{\ell}}$  and thus  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_{\ell}) \neq \emptyset$ . In particular this holds for  $\ell = \infty$ . Henceforth  $\ell$  denotes a prime number.

Let  $M_1 \in \mathbb{Z}^+$  be such that  $C$  extends to a smooth relative curve over  $\mathbb{Z}_{\ell}$  for all  $\ell > M_1$ . Such an  $M_1$  exists for any nice curve  $C_{/\mathbb{Q}}$  by openness of the smooth locus. Since  $C$  is hyperelliptic, we can take  $M_1$  to be the largest prime dividing its minimal discriminant.

Suppose  $\ell > M := \max(M_1, 4g^2 - 1)$ ,  $\ell \neq p$  and  $p \notin \mathbb{Q}_{\ell}^{\times 2}$ . Then the minimal regular model  $C_{/\mathbb{Z}_{\ell}}$  is smooth. We have  $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_{\ell}(\sqrt{p})} \cong C_{/\mathbb{Q}_{\ell}(\sqrt{p})}$ . Since  $\mathbb{Q}_{\ell}(\sqrt{p})/\mathbb{Q}_{\ell}$  is unramified and formation of the minimal regular model commutes with étale base change [L, Prop. 10.1.17] it follows that the minimal regular model  $\mathcal{T}_p(C, \iota)_{/\mathbb{Z}_{\ell}}$  is smooth. By the Riemann hypothesis for curves over a finite field, since  $\ell \geq 4g^2$ , we have  $\mathcal{T}_p(C, \iota)(\mathbb{F}_{\ell}) \neq \emptyset$ , so by Hensel's Lemma we have  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_{\ell}) \neq \emptyset$ .

Suppose  $\ell \leq M$  and  $\ell \neq p$ . If  $\ell = 2$ , then  $p \in \mathbb{Q}_{\ell}^{\times 2}$  because  $p \equiv 1 \pmod{8}$ . If  $\ell$  is odd we require that  $p$  is a quadratic residue modulo  $\ell$ , so again  $p \in \mathbb{Q}_{\ell}^{\times 2}$ . Either way,  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_{\ell}) = C(\mathbb{Q}_{\ell}) \neq \emptyset$ .

Suppose  $\ell = p$ . Let  $P \in C(\overline{\mathbb{Q}})$  be a hyperelliptic branch point. We assume that  $p$  splits completely in  $\mathbb{Q}(P)$ . Then  $P \in C(\mathbb{Q}_p) \cap \mathcal{T}_p(C, \iota)(\mathbb{Q}_p)$ .

All in all we have finitely many conditions on  $p$ , each of the form that  $p$  splits completely in a certain number field. Taking the compositum of these finitely many number fields and its Galois closure, say  $L$ , we see that if  $p$  splits completely in  $L$  then  $\mathcal{T}_p(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ . By (e.g.) the Chebotarev density theorem, this set of primes has positive density.  $\square$

### 2.2. Global.

**Theorem 4.** *(Granville [Gr07, Cor. 1.2]) Assume the ABC conjecture. Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g \geq 3$ . The number of squarefree integers  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)(\mathbb{Q})$  has a point that is not a hyperelliptic branch point is  $\ll_C X^{\frac{1}{g-1} + o(1)} \ll_C X^{2/3}$ .*

**2.3. Local-global.** We now complete the proof of Theorem 2. Let  $(C, \iota)$  be a hyperelliptic curve of genus  $g \geq 3$  without  $\mathbb{Q}$ -rational hyperelliptic branch points, so  $C$  has an affine model of the form  $y^2 = f(x)$  with  $f(x) \in \mathbb{Z}[x]$  of degree  $2g + 2$ , with distinct roots in  $\overline{\mathbb{Q}}$  and no roots in  $\mathbb{Q}$ . Put  $d_0 := f(1)$ . Then  $(1, 1)$  is a  $\mathbb{Q}$ -point on  $d_0 y^2 = f(x)$  and thus on  $\mathcal{T}_{d_0}(C, \iota)$ . The involution  $\iota$  remains

$\mathbb{Q}$ -rational on  $\mathcal{T}_{d_0}(C, \iota)$  (cf. [CS18, §2.1]). We may thus apply Theorem 3 to the hyperelliptic curve  $(\mathcal{T}_{d_0}(C, \iota), \iota)$ , getting a set of primes  $p \equiv 1 \pmod{8}$  of density  $\delta > 0$  such that

$$\mathcal{T}_{pd_0}(C, \iota)_{/\mathbb{Q}} = \mathcal{T}_p(\mathcal{T}_{d_0}(C, \iota), \iota)_{/\mathbb{Q}}$$

has points everywhere locally. By the Prime Number Theorem in Arithmetic Progressions, for at least  $(\frac{\delta}{d_0} + o(1))\frac{X}{\log X}$  squarefree  $d$  with  $|d| \leq X$ , we have  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ . By Theorem 4, we have  $\mathcal{T}_d(C, \iota)(\mathbb{Q}) \neq \emptyset$  for  $\ll X^{2/3}$  squarefree  $d$  with  $|d| \leq X$ . So the number of squarefree  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_C \frac{X}{\log X}$ .

### 3. COUNTING TWISTS WITH ADELIC POINTS

For a hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$ , let

$$\mathfrak{U}_C = \{\text{squarefree } d \in \mathbb{Z} \mid \mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\}$$

be the set of twists of  $C$  having points everywhere locally. For  $X \geq 1$ , put

$$\mathfrak{U}_C(X) = \#\{\mathfrak{U}_C \cap [-X, X]\}.$$

As we saw above, Theorem 3 gives  $\mathfrak{U}_C(X) \gg \frac{X}{\log X}$ .

Recall that a polynomial  $f \in \mathbb{Z}[x]$  is **intersective** if it has roots modulo  $N$  for all  $N \in \mathbb{Z}^+$ , or equivalently, in  $\mathbb{Z}_p$  for all primes  $p$ . We say a polynomial  $f \in \mathbb{Z}[x]$  is **weakly intersective** if the set of prime numbers  $p$  such that  $f$  has a root modulo  $p$  has density 1.

*Remark 5.* Suppose  $f = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$  has degree  $n \geq 2$ , is weakly intersective and has distinct roots in  $\overline{\mathbb{Q}}$ , with discriminant  $\Delta$ . Let  $G$  be the Galois group of  $f$ .

For every prime number  $p \nmid a_n \Delta$ , the partition of  $n$  given by the cycle type of a Frobenius element  $\sigma_p$  at  $p$  coincides with the partition of  $n$  given by the degrees of the irreducible factors of the image of  $f$  in  $\mathbb{Z}/p\mathbb{Z}[x]$ . Since  $f$  is weakly intersective, it follows from the Frobenius Density Theorem (see e.g. [SL96, §3]) that every  $\sigma \in G$  has a fixed point and thus  $f$  has a root mod  $p$  for all  $p \nmid a_n \Delta$ , and thus by Hensel's Lemma it has a root in  $\mathbb{Z}_p$  for all but finitely many  $p$ .

Since every  $\sigma \in G$  has a fixed point, it follows from the Cauchy-Frobenius(–“not Burnside”) Lemma that  $f \in \mathbb{Q}[x]$  is not irreducible.

**Theorem 6.** *Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve. Let  $y^2 = f(x)$  be an affine equation for  $C$  with  $f \in \mathbb{Z}[x]$  squarefree of even degree.*

*a) If  $f$  is weakly intersective then  $\mathfrak{U}_C(X) \gg X$ .*

*b) If  $f$  is not weakly intersective, let  $\beta$  be the density of the set of prime numbers  $p$  such that  $f$  has no root modulo  $p$ , so  $\beta \in (0, 1)$ .<sup>2</sup> Then  $\mathfrak{U}_C(X) \ll \frac{X}{\log^\beta X}$ .*

*Proof.* Let  $\Delta$  be the discriminant of  $f$ .

Step 1: Suppose  $f \in \mathbb{Z}[x]$  is weakly intersective. By Remark 5  $f$  has a root in  $\mathbb{Z}_p$  for all but finitely many  $p$ , and thus the set  $\mathcal{P}$  of prime numbers  $p$  such that  $C(\mathbb{Q}_p) = \emptyset$  is finite. For each  $p \in \mathcal{P}$ , we have  $C_d(\mathbb{Q}_p) \neq \emptyset$  so long as  $d$  lies in the same  $\mathbb{Q}_p$ -adic square class as  $f(1)$ . The set of integers lying in a given  $\mathbb{Q}_p$ -adic square class is a nonempty union of congruence classes modulo  $p^2$  (if  $p > 2$ ) or modulo 16 (if  $p = 2$ ). Applying the Chinese Remainder Theorem, there are  $a, N \in \mathbb{Z}^+$  such that if  $d \equiv a \pmod{N}$  then  $\mathcal{T}_d(C, \iota)(\mathbb{Q}_p) \neq \emptyset$  for all primes  $p$ . Finally, if  $f$  has a real root then  $\mathcal{T}_d(C, \iota)(\mathbb{R}) \neq \emptyset$  for all  $d$ ; otherwise  $\mathcal{T}_d(C, \iota)(\mathbb{R}) \neq \emptyset$  iff  $df(1) > 0$ . Thus  $\mathfrak{U}_C(X) \gg X$ . (The implied constant can be made explicit in terms of  $\Delta$ .)

Step 2: Suppose  $f$  is not weakly intersective. Let  $E'$  be the set of all squarefree integers  $d$  such that for all primes  $p \mid d$ , either  $p \mid 2\Delta$  or  $f$  has a root modulo  $p$ . Let  $E$  be the set of all squarefree integers that do not lie in  $E'$ . Thus for all  $d \in E$ , there is an odd prime  $p \mid d$  such that the image

<sup>2</sup>The polynomial  $f$  has a root modulo every prime  $p$  that splits completely in the splitting field of  $f$ , so  $\beta > 0$ .

of  $f$  in  $\mathbb{Z}/p\mathbb{Z}$  is squarefree and has no root modulo  $p$ . By a result of Sadek [Sa14, Cor. 4.2], this implies that  $\mathcal{T}_d(C)(\mathbb{Q}_p) = \emptyset$ . It follows that

$$\mathfrak{U}_C \subset E'.$$

Let  $E'(X)$  be the number of  $d \in E'$  with  $|d| \leq X$ . Then [Se76, Thm. 2.4] implies that if  $0 < \beta < 1$  then there is  $c > 0$  such that  $E'(X) \sim \frac{cX}{\log^\beta X}$ .  $\square$

We call a hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$  **weakly intersective** if it has a weakly intersective squarefree, integral, even degree defining polynomial.<sup>3</sup> Since no weakly intersective polynomial is irreducible, when genus  $g$  hyperelliptic curves are ordered by height, 0% of them are weakly intersective.

Theorems 2 and 6 immediately imply the following:

**Corollary 7.** *Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g$  without  $\mathbb{Q}$ -rational branch points.*

a) *If  $C$  is weakly intersective and  $g \geq 3$ , then conditionally on ABC, as  $X \rightarrow \infty$  the number of quadratic twists of  $(C, \iota)$  that violate the Hasse Principle is  $\gg X$ .*

b) *If  $C$  is not weakly intersective, then as  $X \rightarrow \infty$ , the number of quadratic twists of  $(C, \iota)$  that violate the Hasse Principle is  $o(X)$ .*

*Example 8.*

a) For any coprime, nonsquare integers  $a, b > 1$ , the polynomial  $(x^2 - a)(x^2 - b)(x^2 - ab)$  is weakly intersective and without rational roots. The polynomial  $(x^2 - 2)(x^2 - 3)(x^2 - 6)$  is not intersective – it has no root in  $\mathbb{Q}_2$ . The polynomial  $(x^2 - 2)(x^2 - 17)(x^2 - 34)$  is intersective.

b) For any  $g \geq 3$ , let  $h(x) \in \mathbb{Z}[x]$  be monic of degree  $2g - 4$ , without rational roots and such that  $g(\pm\sqrt{2}), g(\pm\sqrt{3}), g(\pm\sqrt{6}) \neq 0$ . Then  $C_{/\mathbb{Q}} : y^2 = 2(x^2 - 2)(x^2 - 3)(x^2 - 6)h(x)$  is a weakly intersective hyperelliptic curve of genus  $g \geq 3$  without  $\mathbb{Q}$ -rational branch points. So conditionally on ABC, a positive proportion of the quadratic twists of  $C$  violate the Hasse principle.

c) For every even  $n \geq 2$ , there is a cyclic Galois extension  $F/\mathbb{Q}$  of degree  $n$ , and there is a monic polynomial  $f \in \mathbb{Z}[x]$  such that  $\mathbb{Q}[x]/(f) \cong F$ . The hyperelliptic curve  $C_{/\mathbb{Q}} : y^2 = 2f(x)$  has no  $\mathbb{Q}$ -rational branch points, genus  $\frac{n}{2} - 1$  and  $\mathfrak{U}_C(X) \ll \frac{X}{\log^{1-\frac{1}{n}} X}$ .

#### 4. SOME REMARKS

In [Gr07, Conj. 1.3], Granville conjectures that for all  $g \geq 2$ , if  $f \in \mathbb{Z}[x]$  has degree  $2g + 1$  or  $2g + 2$  and distinct roots in  $\overline{\mathbb{Q}}$ , then there is a constant  $\kappa'_f > 0$  such that the number of squarefree  $d$  with  $|d| \leq X$  such that  $dy^2 = f(x)$  has a  $\mathbb{Q}$ -point that is not a hyperelliptic branch point is  $\sim \kappa'_f X^{\frac{1}{g+1}}$ . The above arguments apply verbatim to show that conditionally on Granville's conjecture, for all  $g \geq 2$ , a hyperelliptic curve  $C_{/\mathbb{Q}}$  has  $\gg_C \frac{X}{\log X}$  twists that violate the Hasse principle iff  $C$  has no  $\mathbb{Q}$ -rational branch points. On the other hand, Vatsal has exhibited a genus one hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$  for which a positive proportion of the quadratic twists have infinitely many rational points [Va98]. Still, it may be true that every hyperelliptic curve of genus 1 without  $\mathbb{Q}$ -rational branch points has infinitely many twists that violate the Hasse Principle.

The present work should be compared to two other works that apply Theorem 1 (or its predecessor [Cl08, Thm. 2]) and Faltings' Theorem to get (unconditional) Hasse Principle violations. Namely, Ozman [Oz12] works with the Atkin-Lehner involution  $w_N$  on a modular curve  $X_0(N)$  for a prime  $N \equiv 1 \pmod{4}$  and Clark-Stankewicz [CS18] works with the Atkin-Lehner involution  $w_D$  on a Shimura curve  $X^D$  for a squarefree  $D > 1$ . Taking  $N > 131$  (resp.  $D > 546$ ) ensures that

<sup>3</sup>By Theorem 6, if one defining polynomial is weakly intersective, then all are.

$X_0(N)/\langle w_N \rangle$  (resp.  $X^D/\langle w_D \rangle$ ) has genus at least 2 and thus finitely many  $\mathbb{Q}$ -points. In each work there is an analysis of  $\mathfrak{U}_C(X)$ , the number of twists up to  $X$  with adelic points. For modular curves  $X_0(N)$ , Ozman shows that  $\mathfrak{U}_C(X) \sim \frac{CX}{\log^\gamma X}$  for a positive constant  $C$  and a  $\gamma \in [0, 1]$  determined in terms of the class group of  $\mathbb{Q}(\sqrt{-N})$  [Oz12, Thm. 5.4] (and cf. [CS18, p. 2841, footnote 5]). In the case of Shimura curves  $X^D$ , Clark-Stankewicz show [CS18, Thm. 8] that

$$\frac{X}{\log^{\alpha_D} X} \ll \mathfrak{U}_{X^D}(X) \ll \frac{X}{\log^{\beta_D} X}$$

for constants  $0 < \beta_D < \alpha_D < 1$  determined in terms of  $D$ , such that  $\lim_{D \rightarrow \infty} \alpha_D - \beta_D = 0$ .

There is some overlap: for a finite nonempty set of  $N$  (resp. of  $D$ ), the pair  $(X_0(N), w_N)$  (resp.  $(X^D, w_D)$ ) is hyperelliptic. E.g. the pair  $(X_0(41), w_{41})$  is hyperelliptic of genus 3 and [Oz12, *loc. cit.*] gives  $\mathfrak{U}_{X_0(41)}(X) \sim \frac{CX}{\log^{\frac{11}{16}} X}$ . Similarly, the pair  $(X^{35}, w_{35})$  is hyperelliptic of genus 3 and [CS18, *loc. cit.*] gives  $\frac{X}{\log^{\frac{15}{16}} X} \ll \mathfrak{U}_{X^{35}}(X) \ll \frac{X}{\log^{\frac{11}{16}} X}$ .

It can be shown that for all hyperelliptic curves  $(C, \iota)_{/\mathbb{Q}}$ , there is  $\alpha = \alpha(C) < 1$  such that  $\mathfrak{U}_C(X) \gg \frac{X}{\log^\alpha X}$ . In fact the same conclusion should hold for any  $(C, \iota)_{/\mathbb{Q}}$  satisfying (T1), (T2) and (T3) in Theorem 1, which amounts to a quantitative strengthening of the local part of this result. We hope to return to this in a future work.

Recent work of Bhargava-Gross-Wang [BGW17] shows that for each fixed  $g \geq 1$ , when genus  $g$  hyperelliptic curves  $(C, \iota)_{/\mathbb{Q}}$  are ordered by height, a positive proportion violate the Hasse Principle. This work is unconditional; moreover, the positive proportion result should be contrasted with Corollary 7b). On the other hand, since all quadratic twists of a hyperelliptic curve induce the same point of the moduli space  $\mathcal{H}_g$  of hyperelliptic curves of genus  $g$ , our result gives (conditionally on ABC) Hasse Principle violations on the largest possible subset of  $\mathcal{H}_g$ .

## 5. ACKNOWLEDGMENTS

We are grateful to Paul Pollack for conveying some insights about weakly intersective polynomials recorded in Remark 5. Funding support for the second author came from National Science Foundation grant DMS-1344994.

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